

Hawking Radiation of a Quantum Black Hole in an Inflationary Universe

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Abstract

The quantum stress-energy tensor of a massless scalar field propagating in the two-dimensional Vaidya-de Sitter metric, which describes a classical model spacetime for a dynamical evaporating black hole in an inflationary universe, is analyzed. We present a possible way to obtain the Hawking radiation terms for the model with arbitrary functions of mass. It is used to see how the expansion of universe will affect the dynamical process of black hole evaporation. The results show that the cosmological inflation has an inclination to depress the black hole evaporation. However, if the cosmological constant is sufficiently large then the back-reaction effect has the inclination to increase the black hole evaporation. We also present a simple method to show that it will always produce a divergent flux of outgoing radiation along the Cauchy horizon where the curvature is a finite value. This means that the Hawking radiation will be very large in there and shall modify the classical spacetime drastically. Therefore the black hole evaporation cannot be discussed self-consistently on the classical Vaidya-type spacetime. Our method can also be applied to analyze the quantum stress-energy tensor in the more general Vaidya-type spacetimes.

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Proper boundary will lead to anti-evaporation of schwarzschild-de Sitter black holes, as corrected in Class. Quantum Grav. 11 (1994) 283.

1 Introduction

Hawking [1] discovered that a black hole formed by collapsing matter will emit particles like a black body with temperature proportional to its surface gravity. As the original calculation is done for a static black hole, which is valid only in the case of the small-evaporation limit, we shall, for self-consistency, take account of the back reaction by solving the semiclassical Einstein equation [2]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi \langle T_{\mu\nu} \rangle, \quad (1.1)$$

where $\langle T_{\mu\nu} \rangle$ is the quantum stress-energy tensor which is a regularized one evaluated in a suitable vacuum [3-5]. As $\langle T_{\mu\nu} \rangle$ is a geometrical object which depends on the geometry of the spacetime, equation (1.1) becomes a highly non-linear set of coupled partial differential equations and solving it is a very difficult task even after several approximations have been adopted [6]. In view of this, as a second-best method, many authors have investigated the black-hole evaporation on the classical spacetime background which represents, in any case, a dynamically evaporating black hole [7-12]. As was found many years ago, the Vaidya metric [13] and Vaidya-Bonnor metric [14] are the most suitable spacetimes describing an evaporating neutral and charged black hole, respectively.

Because the black hole really is a cosmological object, it is worthwhile to examine the effect of cosmological evolution on the process of black-hole evaporation [15-17]. In [15], Davies et. al. investigated the thermodynamics of a black hole in the Reissner-Nordstrom-de Sitter space. In [16] Balbinot et. al. discussed the cosmological boundary condition for a black hole. In [17] Mallett constructed the Vaidya-de Sitter metric

$$ds^2 = - \left(1 - \frac{2M(v)}{r} - \frac{\Lambda}{3}r^2 \right) dv^2 + 2dv dr + r^2(d\theta^2 + \sin^2\theta) d\varphi^2, \quad (1.2)$$

where M is a function of the Eddington-Finkelstein-type advanced time v . After a straightforward tensor calculation one sees that this metric will satisfy the Einstein equation with a cosmological constant Λ and source term which represents the matter streaming radiating inwards; thus, the Vaidya-de Sitter metric could be used to model the classical spacetime of an evaporating black hole in a de Sitter universe. Mallett uses this metric to analyze the process of black-hole evaporation and finds that the hole evaporation will be depressed by the cosmological inflation.

In this paper we will analyze the quantum stress-energy tensor of a massless scalar with an arbitrary function of $M(v)$. Our goal is to see how the universe expanding will affect the process of black-hole evaporation. Before so doing, one should notice that in the Vaidya

spacetime, because of the technical difficulties, we have as yet no ability to exactly evaluate $\langle T_{\mu\nu} \rangle$ except for the linear model [9] in which the mass M is a linear function of the advanced time. For the model in the Vaidya-de Sitter spacetime, the situation becomes worse, because even for the linear model we cannot obtain the exact formula of $\langle T_{\mu\nu} \rangle$. This is unfortunate, since knowing the exact function form of $\langle T_{\mu\nu} \rangle$ would let us see the back-reaction effects on black-hole evaporation immediately.

The present work does not try to evaluate the components of $\langle T_{\mu\nu} \rangle$ exactly. Rather, we will present a simple method to find the Hawking radiation terms among the quantum stress-energy tensor of a massless scalar field propagating in the Vaidya-de Sitter spacetime for the general case with arbitrary functions of $M(v)$. As we know, the Hawking radiation term is a component of the quantum stress-energy tensor, which represents the outgoing energy flow and has a finite value at infinity. From the obtained Hawking radiation terms we can then give a discussion about the back reaction effect on an evaporating black hole immersed in an inflationary universe. Our results show that the cosmological inflation has an inclination to depress the black hole evaporation as claimed in [17]. However, we find the effect of back reaction, i.e., the mass variation, can increase the black-hole evaporation if the cosmological constant is sufficiently large. We also present a simple method to show that it will always produce a divergent flux of outgoing radiation along the Cauchy horizon. This means that the Hawking radiation will be very large there and will modify the classical spacetime drastically. Therefore the black-hole evaporation cannot be discussed self-consistently on the classical Vaidya-type spacetime.

This paper is organized as follows. In section 2, the model is set up and formula for calculating the quantum stress-energy tensor are constructed. In section 3, we present a simple method to find the Hawking radiation terms among the $\langle T_{\mu\nu} \rangle$ and then use the results to discuss the effect of universe expansion on the process of black-hole evaporation. The universal behaviour of producing a divergent flux of outgoing radiation along the Cauchy horizon in the Vaidya spacetime is proved in section 4. Section 5 is devoted to discussion.

2 Model and formalism

We will calculate the quantum stress-energy tensor for a massless scalar field propagating in a two-dimensional spacetime which is that obtained by taking a constant θ and a constant φ in the four-dimensional spacetime described by the Vaidya-de Sitter metric in equation (12). This is because a two-dimensional spacetime is conformally flat and a general method has

been constructed to obtain the renormalized stress-energy tensor of a massless scalar field [3, 18].

To proceed, we first set up the double-null coordinates in the three regions.

Region I

$$ds_1^2 = -(1 - \frac{1}{3}\Lambda r^2)dv^2 + 2dvdr = -(1 - \frac{1}{3}\Lambda r^2)du_1dv, \quad v < 0, \quad (2.1)$$

where we have defined

$$du_1 = dv - (1 - \frac{1}{3}\Lambda r^2)^{-1}, \quad (2.2)$$

and thus

$$u_1 = v - 2\sqrt{3/\Lambda} \coth^{-1}\sqrt{3/\Lambda} r, \quad \text{if } \sqrt{3/\Lambda} r > 1 \quad (2.3a)$$

$$u_1 = v - 2\sqrt{3/\Lambda} \tanh^{-1}\sqrt{3/\Lambda} r, \quad \text{if } \sqrt{3/\Lambda} r < 1. \quad (2.3b)$$

Region II

$$\begin{aligned} ds_2^2 &= -(1 - \frac{2M(v)}{r} - \frac{1}{3}\Lambda r^2)dv^2 + 2dvdr \\ &= -(1 - \frac{2M(v)}{r} - \frac{1}{3}\Lambda r^2) G^{-1}(v, r) du_2 dv \equiv -D(v, r) du_2, dv \end{aligned} \quad (2.4)$$

where we have defined

$$du_2 = G(v, r)dv - 2G(v, r) \left(1 - \frac{2M(v)}{r} - \frac{1}{3}\Lambda r^2\right)^{-1} dr \quad (2.5)$$

and thus G is an integrating factor which satisfies

$$\frac{\partial G}{\partial r} + 2\frac{\partial}{\partial r} \left[G(v, r) \left(1 - \frac{2M(v)}{r} - \frac{1}{3}\Lambda r^2\right)^{-1} \right] = 0. \quad (2.6)$$

it Region III

$$ds_1^2 = -(1 - \frac{1}{3}\Lambda r^2)dv^2 + 2dvdr = -(1 - \frac{1}{3}\Lambda r^2)du_1dv, \quad v > 0, \quad (2.7)$$

where we have defined

$$du_1 = dv - (1 - \frac{1}{3}\Lambda r^2)^{-1}, \quad (2.8)$$

and thus

$$u_1 = v - 2\sqrt{3/\Lambda} \coth^{-1}\sqrt{3/\Lambda} r, \quad \text{if } \sqrt{3/\Lambda} r > 1 \quad (2.9a)$$

$$u_1 = v - 2\sqrt{3/\Lambda} \tanh^{-1}\sqrt{3/\Lambda} r, \quad \text{if } \sqrt{3/\Lambda} r < 1. \quad (2.9b)$$

The above model is initially ($v < 0$) in de Sitter spacetime. Then, at $v = 0$ an imploding δ -functional shell of null fluid with positive mass $M(0) = m_0$ forms a black hole [9]. Next, during the interval $0 < v < v_0$, negative-energy-density null fluid falls into the hole to evaporate it gradually. The evaporation rate depends on $M(v)$ which is assumed to be an arbitrarily continuous function. At $v = v_0$, the black hole completely vanishes and the final geometry ($v > v_0$) is again in de Sitter spacetime. A Penrose diagram for such a model can be depicted by piling up sliced Schwarzschild-de Sitter spacetime (with various mass) so that is continuous. Note that the Penrose diagram of de Sitter and Schwarzschild-de Sitter spacetimes can be found in the paper of Gibbons and Hawking [19]. (See also [20].)

Note that relation (2.3a) is adopted to evaluate the Hawking radiation term (which is a component of the stress tensor at $r \rightarrow \infty$) in region II, as the limiting function $G(v, r \rightarrow \infty)$ is a concern. On the other hand, the relations (2.3b) and (2.9b) are adopted to analyse the divergent behaviour along the Cauchy horizon in region III, as the limiting function $G(v, r \rightarrow 0)$ is the concerning one now. (See section 4.)

To determine the stress-energy tensor $\langle T_{\mu\nu} \rangle$ in region II we need a relation between u_1 , and u_2 . This can be found from the match condition [9]. Matching the coordinate across $v = 0$ gives the following differential equation

$$\begin{aligned} \frac{du_1}{du_2} &= -E(u_1) \sinh^2 \left(\sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right), \\ E(u_1) &= D \left(0, -\sqrt{\frac{3}{\Lambda}} \coth \left(\sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right) \right). \end{aligned} \quad (2.10)$$

Thus

$$ds_2^2 = -D(v, r) E^{-1}(u_1) \operatorname{cosech}^2 \left(\sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right) du_1 dv. \quad (2.11)$$

The two-dimensional stress-energy tensor for a quantized massless scalar field could now be evaluated by relating the null coordinates to a suitable set (\tilde{v}, \tilde{u}) in which the vacuum state is defined [3,4]. In our model, the scalar fields modes for the vacuum have the form $\exp(-i\omega v)$ in the infinite past. However, the metric in equation (2.1) shows that our spacetime will become a de Sitter type and not flat in the asymptotical past. This means that we cannot evaluate the quantum stress tensor in the well known 'Unruh vacuum' [3] which reduces to the Minkowski space in the past. The stress tensor found below, therefore, shall be regarded as that evaluated in the background of the de Sitter universe, i.e. we choose (\tilde{v}, \tilde{u})

$= (v, u_1)$, and our results do not contain that of the particle created in the de Sitter spacetime [19,20]. Keeping this meaning in mind we then, after the typical procedure [9], obtain the renormalized stress-energy tensor of a massless scalar field

$$\begin{aligned} \langle T_{\mu_2\nu_2} \rangle = & \frac{1}{24\pi} \left(\frac{1}{4} D D_{,rr} - \frac{1}{8} (D_{,r})^2 + \sinh^4 \left(\sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right) \right. \\ & \times \left. \left(\frac{1}{2} (E_{,u_1})^2 - E E_{,u_1 u_1} - \sqrt{\frac{\Lambda}{3}} E E_{,u_1} \coth \left(\sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right) \right) - \frac{\Lambda}{6} E^2 \right) \end{aligned} \quad (2.12)$$

In the same way, to determine the stress-energy tensor $\langle T_{\mu_3\nu_3} \rangle$ in region I11 we need the relations between u_1, u_2 and u_3 . These can be found from the match conditions. The coordinates match across $v = 0$ and $v = v_0$ giving the following differential equations

$$\frac{du_1}{du_2} = -F_1(u_1) \cosh^2 \left(\sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right),$$

$$F_1(u_1) = D \left(0, -\sqrt{\frac{3}{\Lambda}} \tanh \left(\sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right) \right). \quad (2.13)$$

$$\frac{du_3}{du_2} = -F_2(u_3) \cosh^2 \left(\sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right),$$

$$F_2(u_3) = D \left(0, -\sqrt{\frac{3}{\Lambda}} \tanh \left(\sqrt{\frac{\Lambda}{3}} \frac{u_3 - v_0}{2} \right) \right). \quad (2.14)$$

Thus

$$ds_2^2 = - \left(1 - \frac{\Lambda}{3} r^2 \right) F_2(u_3) F^{-1}(u_1) \cosh^2 \left(\sqrt{\frac{\Lambda}{3}} \frac{u_3 - v_0}{2} \right) du_1 dv \equiv - \left(1 - \frac{\Lambda}{3} r^2 \right) W du_1 dv. \quad (2.15)$$

Using the same definition of the vacuum state discussed before we then, after the typical procedure [9], obtain the renormalized stress-energy tensor of a massless scalar field

$$\langle T_{\mu_3\nu_3} \rangle = \frac{1}{24\pi} W^{-2} \left[Z_{,u_1} - \frac{1}{2} Z^2 \right], \quad Z \equiv \left[\ln \left(1 - \frac{\Lambda}{3} r^2 \right) W \right]_{,u_1}. \quad (2.16)$$

Note that if $M = 0$ then, as can be easily found $\langle T_{\bar{u}\bar{u}} \rangle_0 = \langle T_{vv} \rangle_0 = -\Lambda/144\pi$ and $\langle T_{\bar{u}v} \rangle_0 = \langle T_{v\bar{u}} \rangle_0 = 0$. Neglecting the constant value of $\langle T_{\mu\nu} \rangle_0$, then, in the region I all components of $\langle T_{\mu\nu} \rangle$ are zero; in region III the only non-zero component of $\langle T_{\mu\nu} \rangle$ is that expressed in equation (2.12); in region II although all components of $\langle T_{\mu\nu} \rangle$ are

non-vanishing, only $\langle T_{u_2 u_2} \rangle$ will give a non-zero value as $r \rightarrow \infty$, This is the Hawking radiation term which represents the outgoing energy flow at infinity.

In the following section we will present a possible way to obtain the Hawking radiation terms in for the model with arbitrary function of mass $M(v)$. From the result the effects of the mass variation on the evaporation of a black hole immersed in de Sitter spacetime are discussed. Then, in section 4 we will show that $\langle T_{u_2 u_2} \rangle$ will always produce a divergent flux of outgoing radiation along the Cauchy horizon.

3 Hawking radiation terms in the stress tensor

From equation (2.12) it is seen that to evaluate $\langle T_{u_2 u_2} \rangle$ we need to know the function of $D(v, r)$, or $G(v, r)$. It is unfortunate that the function $G(v, r)$ which is the solution of equation (2.6) could not be found even for a simple choice of the linear functions $M(v)$. However, as we are only interested in the Hawking radiation terms we do not need to know so much. From the formula of $\langle T_{u_2 u_2} \rangle$ expressed in equation (2.12) we see that, to evaluate the first two terms, only a knowledge of the limiting function $G(v, r \rightarrow \infty)$ is necessary, while the last four terms can be evaluated with merely the limiting function $G(v, r \rightarrow 0)$. We will show below how to find these two limiting functions.

3.1 Radiation in initial stage

We first solve equation (2.6) at initial time $v \rightarrow 0$ to find the function $G(v \rightarrow 0, r)$. Substituting the expansions

$$M(v) \approx m_0 + m_1 v + m_2 v^2, \quad (3.1a)$$

$$G(v, r) \approx g_0(r) + g_1(r)v + g_2(r)v^2, \quad (3.1b)$$

into equation (2.6) we obtain the equation

$$\dot{g}_0 + 2 \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right)^{-1} g_1 + \frac{4m_1}{r} \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right)^{-2} g_0 = 0, \quad (3.2)$$

$$\begin{aligned} \dot{g}_1 + 4 \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right)^{-1} g_2 + \frac{8m_1}{r} \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right)^{-2} g_1 \\ + \frac{16m_1^2}{r} \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right)^{-3} g_0 + \frac{8m_2}{r} \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right)^{-2} g_0 = 0. \end{aligned} \quad (3.3)$$

Choosing $g_0(r) = 1$ we then obtain

$$g_1(r) = -\frac{2m_1}{r} \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right)^{-1}. \quad (3.4a)$$

$$g_2(r) = -\frac{m_1}{2r^2} - \left(\frac{m_0 m_1}{r^3} + \frac{2m_2}{r} - \frac{\Lambda}{3} m_1\right) \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right)^{-1}. \quad (3.4b)$$

Thus we obtain

$$D(v \rightarrow 0, r) = \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right) + \left(\frac{1}{2r^2} - \frac{\Lambda}{2}\right) m_1 v^2. \quad (3.5)$$

(Note that the choice of $g_0(r) = 1$, which implies the above function of $D(v \rightarrow 0, r)$, must be consistent with the function $D(v, r \rightarrow 0)$ solved in section 3.2.)

Substituting the above expression into equation (2.12) we then obtain the Hawking radiation terms in the initial stage

$$\langle T_{u_2 u_2} \rangle \rightarrow \frac{1}{24\pi} (H_0, \tilde{H}_b), \quad (3.6)$$

where

$$\begin{aligned} H_0 = & \frac{\Lambda^2}{9} m_0^2 \sinh^4 \sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \left[2 \left(1 - \operatorname{sech}^2 \sqrt{\frac{\Lambda}{3}} \frac{u_1}{2}\right)^2 + \operatorname{sech}^4 \sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right] \\ & + \frac{\Lambda}{3} m_0 \left[\tanh \sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \left(2 + \operatorname{sech}^2 \sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} + 2 \sinh^2 \sqrt{\frac{\Lambda}{3}} \frac{u_1}{2} \right) \right] \end{aligned} \quad (3.7)$$

$$\tilde{H}_b = \frac{\Lambda^2 m_1}{12} v^2 - \frac{\Lambda}{6} \quad (3.8)$$

Note that as $u_1 < 0$ the value of H_0 is negative. Also, according to the known result, $\dot{M}(v) \sim -M^{-2}(v)$ at the initial stage, thus $m_1 < 0$ and the value of \tilde{H}_b is negative too. We therefore see that the terms coupling the cosmological constant with the mass or mass variation are always negative. These mean that the cosmological inflation has an inclination to depress the black hole evaporation. The conclusions are consistent with [17].

3.2 Radiation at any time

To investigate the black hole evaporation at any time in region II we shall find the limiting function $G(v, r \rightarrow \infty)$ which satisfies equation (2.6). Mathematically, equation (2.6) is a first-order partial differential equation and its solution can be found with the help of the character equation

$$dv = \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right) \frac{dv}{2} = \frac{-r}{4 dM/dv} \left(1 - \frac{2m_0}{r} - \frac{\Lambda}{3} r^2\right) d \ln G. \quad (3.9)$$

When $r \rightarrow \infty$, the first equation gives a simple solution

$$\frac{6}{\Lambda r} - v \approx c_1, \quad (3.10)$$

where c_1 is an integration constant. With this relation the second equation becomes (the approximation $r \rightarrow \infty$ has been taken)

$$d \ln G^{-1} = \frac{-\dot{M} \Lambda^2}{36} (v + c_1)^3 dv. \quad (3.11)$$

The above equation can now be integrated formally. Again, replacing the c_1 by the relation equation (3.10) and taking the limiting $r \rightarrow \infty$ we finally obtain

$$G^{-1}(v, r) \approx c_2 e^{\Lambda^2 A/36} \left[1 + \frac{\Lambda}{2r} B + \frac{1}{r} \left(\frac{\Lambda^2}{8} B^2 - C \right) \right], \quad (3.12)$$

where A , B and C are defined by Note that although the integration constant c_2 may be an arbitrary one it

$$A \equiv - \int_0^v \dot{M}(\tilde{v})(v - \tilde{v})^3 d\tilde{v} \quad (3.13a)$$

$$B \equiv - \int_0^v \dot{M}(\tilde{v})(v - \tilde{v})^2 d\tilde{v} \quad (3.13b)$$

$$C \equiv -3 \int_0^v \dot{M}(\tilde{v})(v - \tilde{v}) d\tilde{v}. \quad (3.13c)$$

Note that although the integration constant c_2 may be an arbitrary one it must be chosen to be consistent with the function $G(v \rightarrow 0, r)$ found in section 3.1. Therefore c_2 shall be chosen as one.

Using the above obtained limiting functions $G(v, r \rightarrow \infty)$ and $G(v \rightarrow 0, r)$ we finally from equation (2.12) obtain the Hawking radiation terms

$$\langle T_{u_2 u_2} \rangle \rightarrow \frac{1}{24\pi} (H_0 + H_b), \quad \text{as } r \rightarrow \infty. \quad (3.14)$$

where H_0 being defined in equation (3.7) does not depend on the advance time v , and H_b showing the effect of back reaction is defined by

$$H_b \equiv \left(\frac{\Lambda^4}{288} B^2 - \frac{\Lambda^2}{18} C - \frac{\Lambda}{6} \right) e^{\Lambda^2 A/18}. \quad (3.15)$$

with A , B and C defined in equation (3.13).

From equation (3.14) we see that when the cosmological constant Λ is small then $H_b = -\Lambda/6 - \Lambda^2 C/18$. Because C is positive (see equation (3.13b)) the back-reaction effect will depress the black-hole evaporation. (If both Λ and $v \rightarrow 0$ then the above results give the

value of $\langle T_{u_2 u_2} \rangle$ as described by equation (3.6).) Combining the fact that H_0 defined in equation (3.7) is negative, we thus see that the terms coupling the cosmological constant with the mass or mass variation are always negative. This means that the cosmological inflation has an inclination to depress the black hole evaporation, as claimed in [17]. On the other hand, if the cosmological constant Λ becomes very large then it is apparent that H_b , will become a positive function. This means that a time-dependent mass term entering the Vaidya-de Sitter metric will increase the Hawking radiation if the cosmological constant is sufficiently large, as claimed in the introduction. (This conclusion does not agree with [17].) Notice that H_0 , being always positive, is a function of Λ , $M(0)$ and u_1 ; while H_b , being positive if A is very large, is a function of Λ , $\dot{M}(v)$ and v .

Finally we shall make two remarks. (i) One may wonder why the property of increasing the black-hole evaporation by the back-reaction effect when the cosmological constant is large does not show in section 3.1. The reason is that the expansion at $v = 0$, i.e., in initial stage, will correspond to the expansion about $\Lambda = 0$, according to the formulation in section 3.1. This may be seen from the fact that taking more expansion terms in equation (3.1) will let the function of $D(v \rightarrow 0, r)$ in equation (3.5) contain the terms proportional to λ, \dots etc. (ii) Once $\lambda = 0$ was taken, then nothing would be obtained. One may wonder why the result [12] for the model in the Vaidya spacetime will not be obtained in this limit. The reason is that we adopt equation (2.3a) rather than equation (2.3b) during the formulation, as we are interested in the Hawking radiation term which is defined at infinity, then equation (2.3a) does not define itself at $\lambda = 0$.

4 Divergence of the stress tensor along the Cauchy horizon

The analyses The analyses for the linear model [9] in the Vaidya or Vaidya-Bonnor spacetime have found that the stress-energy tensor in region III will be divergent as the Cauchy horizon is approached, i.e., $u_3 \rightarrow v_0$. This divergent property has been shown to be very general. In this section we present a simple way to prove that the quadratic divergence of the quantum stress-energy is universal for the model described in the Vaidya-de Sitter spacetime.

From equation (2.16) we see that, as we are only interested in the stress-energy tensor as the Cauchy horizon is approached, i.e., $u_3 \rightarrow v_0$, the only function to be found is the limiting form of $F_2(u_3 \rightarrow u_o) = D(v_0, -\frac{1}{2}(u_3 - v_0) \rightarrow 0)$ and its corresponding limiting function of

$F_l(u_l)$. We can find these two functions easily as shown below.

In the limit of $r \rightarrow 0$ equation (2.6) becomes

$$\frac{\partial G}{\partial r} \approx r \frac{\partial}{\partial v} (G/M), \quad r \rightarrow 0, \quad (4.1)$$

and a simple solution we have is

$$G(v_0, r \rightarrow 0) \approx c(1 + r^2). \quad (4.2)$$

where c is an integration constant. Choosing a suitable value of c we have the solution

$$F_2(u_3 \rightarrow v_0) = D(v_0, -\frac{1}{2}(u_3 - v_0) \rightarrow 0) \approx \frac{1}{u_3 - v_0}. \quad (4.3)$$

Also, using equations (2.13) and (2.14) we have the relation

$$\begin{aligned} \int^{u_1} F_1(\tilde{u}_1) \cosh^2 \left(\sqrt{\frac{\Lambda}{3}} \frac{\tilde{u}_1}{2} \right) d\tilde{u}_1 &= \int^{u_1} F_2^{-1}(\tilde{u}_3) \operatorname{sech}^2 \left(\sqrt{\frac{\Lambda}{3}} \frac{\tilde{u}_3 - v_0}{2} \right) d\tilde{u}_3 \\ &\approx \int^{u_1} (\tilde{u}_3 - v_0) d\tilde{u}_3, \quad \text{as } u_3 \rightarrow v_0, \end{aligned} \quad (4.4)$$

which tells us that as $u_3 \rightarrow v_0$ the value of u_1 approaches a value (say) s_0 and the integrated function is finite. Thus we have the approximation

$$F_1(u_1) \cosh^2 \left(\sqrt{\frac{\Lambda}{3}} \frac{\tilde{u}_1}{2} \right) \approx a + b(u_1 - s_0)^\alpha, \quad \alpha > -1, \quad \text{as } u_1 \rightarrow s_0, \quad (4.5)$$

Substituting the above obtained limit functions $F_2(u_3 \rightarrow v_0)$ and $F_1(u_2 \rightarrow s_0)$ into equation (2.16) and through some careful analyses we finally obtain the results

$$\langle T_{u_3 u_3} \rangle \rightarrow \frac{1}{16\pi(u_3 - v_0)^2}, \quad \text{as } u_3 \rightarrow v_0, \quad (4.6)$$

The above relation shows that the divergence along the Cauchy horizon is a universal property. However, a divergence appearing at the Cauchy horizon where the curvature has a finite value seems physically serious. In fact, this tells us that the Hawking effect will be very large there and will modify the classical spacetime drastically. Therefore, the evaporation of a quantum black hole cannot be discussed self-consistently on the classical Vaidya-de Sitter spacetime.

5 Conclusion

Several years ago, Davies et al [15] investigated the thermodynamics of a black hole in the Reissner-Nordstrom-de Sitter space. They used the black-hole temperature and cosmological horizon temperature to discuss how the thermal radiation will flow between the hole and universe. However, as is well known, a rigorous description of black-hole evaporation requires the back-reaction effect to be seriously taken into account. This then leads Mallett [17] to construct the Vaidya-de Sitter metric to model the classical spacetime of an evaporating black hole in a de Sitter universe. Mallett uses this metric to analyse the dynamical behaviour of a black hole immersed in an inflationary universe and finds that the process of black-hole evaporation will be depressed by the cosmological inflation.

An apparent way to analyze the back-reaction effect on an evaporating black hole immersed in an inflationary universe is to analyze the quantum stress-energy tensor of a massless scalar field propagating in the two-dimensional Vaidya-de Sitter metric. At first sight, because, to our knowledge, nobody has the ability to solve equation (2.6) except for the step model in which $M(v)$ is a constant, it seems that such a way is hopeless. In this paper, we do not try to solve equation (2.6); rather, we have succeeded in presenting a simple method to find the Hawking radiation term, which is a component of the quantum stress-energy tensor and represents the outgoing energy flow at infinity. From the obtained Hawking radiation terms we then give a discussion about the back-reaction effect on an evaporating black hole immersed in an inflationary universe. Our results show that the cosmological inflation has an inclination to depress the black hole evaporation as claimed in [17]. However, we find that the effect of back-reaction, i.e., the mass variation, can increase the black-hole evaporation if the cosmological constant is sufficiently large. We also have presented a simple method to show that it will always produce a divergent flux of outgoing radiation along the Cauchy horizon where the curvature is a finite value. This means that the Hawking radiation will be very large there and will modify the classical spacetime drastically. Therefore we conclude that the black-hole evaporation cannot be discussed self-consistently on the classical Vaidya-type spacetime.

Finally we want to make a remark. As the prescription presented in this paper is very general and simple, it will be (at least, we hope) helpful to analyze the dynamical behavior of a black hole in more general, more complex and relativistic Vaidya-type spacetimes.

More references can be found in [22-24].

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